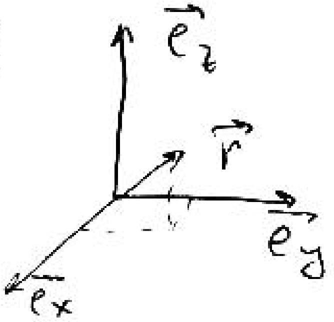
Кинематика

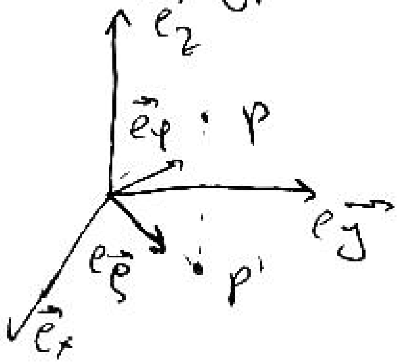
1) а)



орты: $\vec e_x, \vec e_y, \vec e_z$ коорд: x, y, z.

$$\vec r = x\vec e_x + y\vec e_y + z\vec e_z$$
$$\dot{\vec r} = \dot x\vec e_x + \dot y\vec e_y + \dot z\vec e_z$$

б) цилиндр

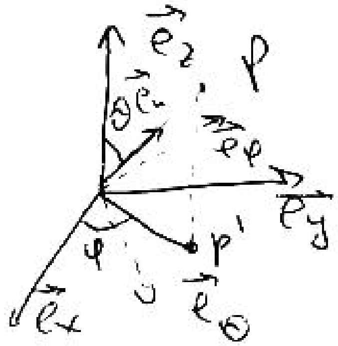


$\vec e_\rho, \vec e_\varphi, \vec e_z$ ρ, φ, z

$x = \rho\cos\varphi$
$y = \rho\sin\varphi$
$z = z$.

$$\vec r = \rho\vec e_\rho + z\vec e_z$$
$$\dot{\vec r} = \dot\rho\vec e_\rho + \rho\dot\varphi\vec e_\varphi + \dot z\vec e_z$$

в) сферич



$\vec e_r, \vec e_\theta, \vec e_\varphi$ r, θ, φ

$x = r\sin\theta\cos\varphi$
$y = r\sin\theta\sin\varphi$
$z = r\cos\theta$.

$$\vec r = r\vec e_r$$
$$\dot{\vec r} = \dot r\vec e_r + r\dot\theta\vec e_\theta + r\sin\theta\dot\varphi\vec e_\varphi$$

2) $I_{ij} = \sum_k m_k(x_l^2\delta_{ij} - x_i x_j)$ $(i, j, l = 1, 2, 3)$

$$I_{ij} = \int(x_l^2\delta_{ij} - x_i x_j)\rho\, dV$$

$$I_{ij} = \int\rho\, dV\begin{pmatrix} y^2+z^2 & -xy & -xz \\ -xy & x^2+z^2 & -yz \\ -xz & -yz & x^2+y^2 \end{pmatrix}$$

Стержень: $I = \frac{Ml^2}{12}$ (ось через середину)

Шар: $I = \frac{2}{5}MR^2$

Диск: $I = \frac{MR^2}{2}$ (через ось диска)

3) $T = \frac{M v_c^2}{2} + \frac{1}{2} I_{ij} \Omega_i \Omega_j$

Механика Лагранжа

1) $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = Q_j^{\partial}$ $\quad L = T - U$ - ф. лагр.

$Q_j^{\partial} = \vec{F}_i^{\partial} \frac{d r_i}{d q_j}$ - обобщ. диссипат. силы.

2) $p_j^{об} = \frac{\partial L}{\partial \dot{q}_j}$ Если $Q_j^{\partial} = 0$ и $\frac{\partial L}{\partial q_j} = 0$ (q_j - циклич. коорд), то $p_j^{об} = const$.

3) $E = \sum \frac{\partial L}{\partial \dot{q}_j} \dot{q}_j - L$

Если $Q_j^{\partial} = 0$ и $\frac{\partial L}{\partial t} = 0$, то $E^{об} = const$.

4) а) $L = \frac{m\dot{\vec{r}}^2}{2} - U(\vec{r}, t)$ Нерелятив.

б) $L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - U(x, y, z, t)$

в) $L = \frac{m}{2}(\dot{\rho}^2 + (\rho\dot{\varphi})^2 + \dot{z}^2) - U(\rho, \varphi, z, t)$

г) $L = \frac{m}{2}(\dot{r}^2 + (r\dot{\theta})^2 + (r\sin\theta\dot{\varphi})^2) - U(r, \theta, \varphi, t)$

$\vec{F} = -\operatorname{grad} U(\vec{r}, t)$

Релятив.

а) $L = -mc^2\sqrt{1 - \frac{\dot{\vec{r}}^2}{c^2}} - U(\vec{r}, t)$

б) $L = -mc^2\sqrt{1 - \frac{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}{c^2}} - U(x, y, z, t)$

в) $L = -mc^2\sqrt{1 - \frac{\dot{\rho}^2 + (\rho\dot{\varphi})^2 + \dot{z}^2}{c^2}} - U(\rho, \varphi, z, t)$

г) $L = -mc^2\sqrt{1 - \frac{\dot{r}^2 + (r\dot{\theta})^2 + (r\sin\theta\dot{\varphi})^2}{c^2}} - U(r, \theta, \varphi, t)$

5) а) $L = \frac{m\dot{\vec{r}}^2}{2} - e\varphi + \frac{e}{c}(\vec{A}, \dot{\vec{r}})$ нерелятив.

б) $L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - e\varphi + \frac{e}{c}(\vec{A}, \dot{x}\vec{e}_x + \dot{y}\vec{e}_y + \dot{z}\vec{e}_z)$

в) $L = \frac{m}{2}(\dot{\rho}^2 + (\rho\dot{\varphi})^2 + \dot{z}^2) - e\varphi + \frac{e}{c}(\vec{A}, \dot{\rho}\vec{e}_\rho + \rho\dot{\varphi}\vec{e}_\varphi + \dot{z}\vec{e}_z)$

г) $L = \frac{m}{2}(\dot{r}^2 + (r\dot{\theta})^2 + (r\sin\theta\dot{\varphi})^2) - e\varphi + \frac{e}{c}(\vec{A}, \dot{r}\vec{e}_r + r\dot{\theta}\vec{e}_\theta + r\sin\theta\dot{\varphi}\vec{e}_\varphi)$

Релятив.

а) $L = -mc^2\sqrt{1 - \frac{\dot{\vec{r}}^2}{c^2}} - e\varphi + \frac{e}{c}(\vec{A}, \dot{\vec{r}})$

б) $L = -mc^2\sqrt{1 - \frac{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}{c^2}} - e\varphi + \frac{e}{c}(\vec{A}, \dot{x}\vec{e}_x + \dot{y}\vec{e}_y + \dot{z}\vec{e}_z)$

в) $L = -mc^2\sqrt{1 - \frac{\dot{\rho}^2 + (\rho\dot{\varphi})^2 + \dot{z}^2}{c^2}} - e\varphi + \frac{e}{c}(\vec{A}, \dot{\rho}\vec{e}_\rho + \rho\dot{\varphi}\vec{e}_\varphi + \dot{z}\vec{e}_z)$

г) $L = -mc^2\sqrt{1 - \frac{\dot{r}^2 + (r\dot{\theta})^2 + (r\sin\theta\dot{\varphi})^2}{c^2}} - e\varphi + \frac{e}{c}(\vec{A}, \dot{r}\vec{e}_r + r\dot{\theta}\vec{e}_\theta + r\sin\theta\dot{\varphi}\vec{e}_\varphi)$

~~Механика Гамильтона~~

~~1) $H = E^{об}(q_i, \dot{q}_i \to p_i, t)$~~

Вект. и скалярн. потенц.

а) $\vec{H} = \mathrm{rot}\,\vec{A}$ $\quad E = -\mathrm{grad}\,\varphi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}$

б) $\vec{E} = E_0\vec{n}$: $\vec{A} = 0$ $\quad \varphi = -(\vec{E}, \vec{r})$

в) $\vec{H} = H_0\vec{e}_z$: $\varphi = 0$

$A = \frac{1}{2}H_0(-y\vec{e}_x + x\vec{e}_y)$ - дек.

$A = \frac{1}{2}H_0\rho\vec{e}_\varphi$ - цил.

$A = \frac{1}{2}H_0 r\sin\theta\,\vec{e}_\varphi$ - сферич.

Механика Гамильтона.

1) $H = E^{об}(q_i, \dot{q}_i \xrightarrow{*} p_i, t)$ $\quad * \; p_i = \frac{\partial L}{\partial \dot{q}_i}$, $E_{об} = \sum_j \frac{\partial L}{\partial \dot{q}_j} \dot{q}_j - L$

Интегр. движения:

1) q_i^* - циклич $\left(\frac{\partial H}{\partial q_i^*} = 0\right)$, $Q_i^d = 0 \Rightarrow \underline{p_i^* = p_0}$

2) $Q^d = 0$, $\frac{\partial H}{\partial t} = 0 \Rightarrow \underline{H = E_0}$

2) Нерелятив

а) $H = \frac{\vec{p}^2}{2m} + u(\vec{r}, t)$

б) $H = \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2) + u(x, y, z, t)$

в) $H = \frac{1}{2m}\left(p_\rho^2 + \left(\frac{p_\varphi}{\rho}\right)^2 + p_z^2\right) + u(\rho, \varphi, z, t)$

г) $H = \frac{1}{2m}\left(p_r^2 + \left(\frac{p_\theta}{r}\right)^2 + \left(\frac{p_\varphi}{r\sin\theta}\right)^2\right) + u(r, \theta, \varphi, t)$

Релятив.

а) $H = \sqrt{m^2c^4 + \vec{p}^2c^2} + u(\vec{r}, t)$

б) $H = \sqrt{m^2c^4 + (p_x^2 + p_y^2 + p_z^2)c^2} + u\,\cancel{(\vec{r}, t)}\,(x, y, z, t)$

в) $H = \cancel{\frac{1}{2m}}\sqrt{m^2c^4 + \left(p_\rho^2 + \left(\frac{p_\varphi}{\rho}\right)^2 + p_z^2\right)c^2} + u(\rho, \varphi, z, t)$

г) $H = \sqrt{m^2c^4 + \left(p_r^2 + \left(\frac{p_\theta}{r}\right)^2 + \left(\frac{p_\varphi}{r\sin\theta}\right)^2\right)c^2} + u(r, \theta, \varphi, t)$

3) Нерелятив:

а) $H = \dfrac{(\vec{p} - \frac{e}{c}\vec{A})^2}{2m} + e\varphi$

б) $H = \dfrac{1}{2m}\left[(p_x - \frac{e}{c}A_x)^2 + (p_y - \frac{e}{c}A_y)^2 + (p_z - \frac{e}{c}A_z)^2\right] + e\varphi(x,y,z,t)$

в) $H = \dfrac{1}{2m}\left[(p_\rho - \frac{e}{c}A_\rho)^2 + (\frac{p_\varphi}{\rho} - \frac{e}{c}A_\varphi)^2 + (p_z - \frac{e}{c}A_z)^2\right] + e\varphi(\rho,\varphi,z,t)$

г) $H = \dfrac{1}{2m}\left[(p_r - \frac{e}{c}A_r)^2 + (\frac{p_\theta}{r} - \frac{e}{c}A_\theta)^2 + (\frac{p_\varphi}{r\sin\theta} - \frac{e}{c}A_\varphi)^2\right] + e\varphi(r,\theta,\varphi,t)$

Релятив

б) $H = \sqrt{m^2c^4 + (p_x - \frac{e}{c}A_x)^2c^2 + (p_y - \frac{e}{c}A_y)^2c^2 + (p_z - \frac{e}{c}A_z)^2c^2} + e\varphi(x,y,z,t)$

а) $H = \sqrt{m^2c^4 + (\vec{p} - \frac{e}{c}\vec{A})^2c^2} + e\varphi$

в) $H = \sqrt{m^2c^4 + (p_\rho - \frac{e}{c}A_\rho)^2c^2 + (\frac{p_\varphi}{\rho} - \frac{e}{c}A_\varphi)^2c^2 + (p_z - \frac{e}{c}A_z)^2c^2} + e\varphi(\rho,\varphi,z,t)$

г) $H = \sqrt{m^2c^4 + (p_r - \frac{e}{c}A_r)^2c^2 + (\frac{p_\theta}{r} - \frac{e}{c}A_\theta)^2c^2 + (\frac{p_\varphi}{r\sin\theta} - \frac{e}{c}A_\theta)^2c^2} + e\varphi(r,\theta,\varphi,t)$

4) $\begin{cases} \dot{q}_i = \dfrac{\partial H}{\partial p_i} \\ \dot{p}_i = -\dfrac{\partial H}{\partial q_i} + Q_i^d \end{cases}$

5) $\{f,g\} = \sum_\alpha \left\{ \dfrac{\partial f}{\partial q_\alpha}\dfrac{\partial g}{\partial p_\alpha} - \dfrac{\partial f}{\partial p_\alpha}\dfrac{\partial g}{\partial q_\alpha} \right\}$

$\{q_i, q_j\} = 0 \quad \{p_i, p_j\} = 0 \quad \{q_i, p_j\} = \delta_{ij}$ – фунд. скобки.

6) Если $H(F(q_1, \dots q_l, p_1 \dots p_l), q_{l+1} \dots q_s, p_{l+1} \dots p_s, t)$

то $F(q_1 \dots q_l, p_1 \dots p_l)$ – интегр движения

$\dfrac{\partial F}{\partial t}$ (част. произв) $+$

$\{F, H\} = 0$

5

7) I а) $H(q_i, p_i, t)$

б) Ур. Г-я в явном виде.

II полный интеграл ур-я Г-Я ≡ решения ур-я

$\frac{\partial F}{\partial t} + H\left(q_i, p_i = \frac{\partial F}{\partial q_i}, t\right) = 0$, зависящее от S констант.

$F = F(q_1 \dots q_s, \alpha_1 \dots \alpha_s, t) + \alpha_{s+1}$

III Теорема Якоби $Q_i = \frac{\partial F}{\partial p_i}$

$Q_1 = \frac{\partial F}{\partial p_1}$ $\frac{\partial F}{\partial \alpha_i} = p_i \Rightarrow q_i = q_i(\alpha_1 \dots \alpha_s, p_1 \dots p_s, t)$

α_i, β_i - из нач. услов

Записать ф-ии Лагранжа, Гамильтона и Г-Я.

#1 Нерелятив:

1) $L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz, \quad \vec{g} = -\vec{g}e_z$

$H = \frac{p_x^2 + p_y^2 + p_z^2}{2m} + mgz \qquad \frac{\partial F}{\partial t} + \frac{1}{2m}\left(\left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2 + \left(\frac{\partial F}{\partial z}\right)^2\right) + mgz = 0$

2) $L = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + e(\vec{E}, \vec{r}) \quad (\vec{E} = E_0\vec{r})$

$H = \frac{1}{2m}\left(p_x^2 + p_y^2 + p_z^2\right) - e[\vec{E}, \vec{r})$

$\frac{\partial F}{\partial t} + \frac{1}{2m}\left(\left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2 + \left(\frac{\partial F}{\partial z}\right)^2\right) - e(\vec{E}, \vec{r}) = 0$

3) $L = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + \frac{eH_0}{2c}(x\dot{y} - y\dot{x}) \quad (\vec{H} = H_0\vec{e}_z)$

$H = \frac{1}{2m}\left[\left(p_x + \frac{e}{2c}yH_0\right)^2 + \left(p_y - \frac{e}{2c}xH_0\right)^2 + p_z^2\right]$

$\frac{\partial F}{\partial t} + \frac{1}{2m}\left[\left(\frac{\partial F}{\partial x} + \frac{eH_0}{2c}y\right)^2 + \left(\frac{\partial F}{\partial y} - \frac{eH_0}{2c}x\right)^2 + \left(\frac{\partial F}{\partial z}\right)^2\right) = 0$

4) $L = \frac{m}{2}R^2\left[\dot{\theta}^2 + (\sin\theta\ \dot{\varphi})^2\right) - mgR\cos\theta$

$H = \frac{1}{2mR^2}\left(p_\theta + \left(\frac{p_\varphi}{\sin\theta}\right)^2\right) + mgR\cos\theta$

$\frac{\partial F}{\partial t} + \frac{1}{2mR^2}\left(\left(\frac{\partial F}{\partial \theta}\right)^2 + \left(\frac{\partial F}{\partial \varphi}\right)^2 \frac{1}{\sin^2\theta}\right) + mgR\cos\theta = 0$

5) $L = \frac{m}{2}\left(\dot{r}^2 + (r\dot{\theta})^2 + [r\sin\theta\dot{\varphi}]^2\right) - U(r)$

$H = \frac{1}{2m}\left(p_r^2 + \left(\frac{p_\theta}{r}\right)^2 + \left(\frac{p_\varphi}{r\sin\theta}\right)^2\right) + U(r)$

$\frac{\partial F}{\partial t} + \frac{1}{2m}\left(\left(\frac{\partial F}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial F}{\partial \theta}\right)^2 + \frac{1}{r^2\sin^2\theta}\left(\frac{\partial F}{\partial \varphi}\right)^2\right) + U(r) = 0$

Если силы не действ. то случай плоский: 3-их перем. нет.

6) $L = \frac{m}{2}\dot{x}^2 - \frac{m\omega^2}{2}x^2$

$H = \frac{p_x^2}{2m} + \frac{m\omega^2 x^2}{2}$

$\frac{\partial F}{\partial t} + \frac{1}{2m}\left(\frac{\partial F}{\partial x}\right)^2 + \frac{m\omega^2 x^2}{2} = 0$

Решение 6.

1) $L = -mc^2\sqrt{1 - \frac{\dot{x}^2+\dot{y}^2+\dot{z}^2}{2}} - mgz$

$H = \sqrt{m^2c^4 + (p_x^2+p_y^2+p_z^2)c^2} + mgz$

$\frac{\partial F}{\partial t} + \sqrt{m^2c^4 + \left(\left(\frac{\partial H}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2 + \left(\frac{\partial F}{\partial z}\right)^2\right)c^2} + mgz.$

2) $L = -mc^2\sqrt{1 - \frac{\dot{x}^2+\dot{y}^2+\dot{z}^2}{2}} + e(\vec{E}\,\vec{r})$

$H = \sqrt{m^2c^4 + (p_x^2+p_y^2+p_z^2)c^2} - e(\vec{E}\,\vec{r})$

$\frac{\partial F}{\partial t} + \sqrt{m^2c^4 + \left(\left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2 + \left(\frac{\partial F}{\partial z}\right)^2\right)c^2} - e(\vec{E}\cdot\vec{r}) = 0$

3) $L = -mc^2\sqrt{1 - \frac{\dot{x}^2+\dot{y}^2+\dot{z}^2}{2}} + \frac{eH_0}{2c}(x\dot{y} - y\dot{x})$

$H = \sqrt{m^2c^4 + \left(\left(p_x + \frac{e}{2c}yH_0\right)^2 + \left(p_y - \frac{e}{2c}xH_0\right)^2 + p_z^2\right)c^2}$

$\frac{\partial F}{\partial t} + \sqrt{m^2c^4 + \left(\left(\frac{\partial F}{\partial x} + \frac{eH_0}{2c}y\right)^2 + \left(\frac{\partial F}{\partial y} - \frac{eH_0}{2c}x\right)^2 + \left(\frac{\partial F}{\partial z}\right)^2\right)c^2} = 0$

4) $L = -mc^2\sqrt{1 - \frac{R^2(\dot{\theta}^2 + (\sin\theta\,\dot{\varphi})^2)}{2}} - mgR\cos\theta$

$H = \sqrt{m^2c^4 + \frac{1}{R^2}\left(p_\theta^2 + \left(\frac{p_\varphi}{\sin\theta}\right)^2\right)c^2} + mgR\cos\theta$

$\frac{\partial F}{\partial t} + \sqrt{m^2c^4 + \frac{c^2}{R^2}\left(\left(\frac{\partial F}{\partial \theta}\right)^2 + \left(\frac{\partial F}{\partial \varphi}\right)^2\frac{1}{\sin^2\theta}\right)} + mgR\cos\theta = 0$

5) $L = -mc^2\sqrt{1 - \frac{\dot{r}^2 + (r\dot{\theta})^2 + (r\sin\theta\,\dot{\varphi})^2}{2}} - U(r)$

$H = \sqrt{m^2c^4 + \left(p_r^2 + \frac{p_\theta^2}{r^2} + \left(\frac{p_\varphi}{r\sin\theta}\right)^2\right)c^2} + U(r)$

$\frac{\partial F}{\partial t} + \sqrt{m^2c^4 + c^2\left(\frac{\partial F}{\partial r} + \frac{1}{r^2}\left(\frac{\partial F}{\partial \theta}\right)^2 + \frac{1}{r^2\sin^2\theta}\left(\frac{\partial F}{\partial \varphi}\right)^2\right)} + U(r) = 0$

6) $L = -mc^2\sqrt{1 - \frac{\dot{x}^2}{2}} - \frac{m\omega^2}{2}x^2$ $\qquad H = \sqrt{m^2c^4 + p_x^2c^2} + \frac{m\omega^2}{2}x^2$

$\frac{\partial F}{\partial t} + \sqrt{m^2c^4 + c^2\left(\frac{\partial F}{\partial x}\right)^2} + \frac{m\omega^2x^2}{2} = 0$